

Derivation of fluid dynamics from kinetic theory with the 14-moment approximation

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Abstract. We review the traditional derivation of the fluid-dynamical equations from kinetic theory according to Israel and Stewart. We show that their procedure to close the fluid-dynamical equations of motion is not unique. Their approach contains two approximations, the first being the so-called 14-moment approximation to truncate the single-particle distribution function. The second consists in the choice of equations of motion for the dissipative currents. Israel and Stewart used the second moment of the Boltzmann equation, but this is not the only possible choice. In fact, there are infinitely many moments of the Boltzmann equation which can serve as equations of motion for the dissipative currents. All resulting equations of motion have the same form, but the transport coefficients are different in each case.

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1 Introduction

Fluid dynamics is an effective theory to describe the long-wavelength, low-frequency dynamics of macroscopic systems. In non-relativistic systems, the Navier-Stokes equations are able to describe a wide variety of fluids, from weakly interacting gases, such as air, to liquids, such as water. On the other hand, the theory of relativistic dissipative fluid dynamics has not yet been completely established and remains a topic of intense investigation. For dilute systems, the derivation of fluid dynamics can be investigated starting from the relativistic Boltzmann equation.

Chapman-Enskog theory [1] is a well-known approach to derive fluid-dynamical equations from the Boltzmann equation. In this approach, the single-particle distribution function which is the solution of the Boltzmann equation is expressed in terms of an expansion in gradients of the primary fluid-dynamical variables, i.e., chemical potential, temperature, and velocity, each term containing a different power or order of derivatives. This leads to a series in powers of the Knudsen number, $\text{Kn} = \lambda/L$, the ratio of the mean-free path of the particles, λ , and a characteristic macroscopic length, L . As is well-known, to zeroth order this method leads to the equations of ideal fluid dynamics. To first order in Knudsen number one obtains the Navier-Stokes equations of fluid dynamics. To second and higher order in Knudsen number one obtains the Burnett and super-Burnett equations. However, rela-

tivistic Navier-Stokes theory, as well as any higher-order truncation of the relativistic Chapman-Enskog expansion is unstable and, consequently, unsuitable to describe any relativistic fluid existing in nature [2,3,4].

The source of such an instability is well understood in the relativistic case: it comes from the acausality of Navier-Stokes theory [3,4]. Therefore, a consistent and stable theory of relativistic fluid dynamics must also be causal. Causal fluid-dynamical equations were first derived from kinetic theory by H. Grad [5], for non-relativistic systems via the method of moments. In Grad's original work, the single-particle distribution function is expanded around local equilibrium in terms of a complete set of Hermite polynomials [6]. Fluid dynamics is obtained by explicitly truncating this expansion, expressing the distribution function in terms of only 13 moments: the velocity field, the temperature, the chemical potential, the heat current, and the shear-stress tensor. Due to this truncation scheme the method became known as the 13-moment approximation. In non-relativistic systems, the correction to the equilibrium pressure, the bulk viscous pressure, vanishes and is not included in the 13 variables.

The generalization of Grad's moment method to relativistic systems is non-trivial. One reason is the lack of a suitable set of orthogonal polynomials to replace the Hermite polynomials [7,8,9]. Despite this problem, relativistic generalizations have been given by several authors [10,11,12,13,14]. One of the most well-known works on

this topic was done by Israel and Stewart (IS) [8,9,15]. In this approach, the single-particle distribution function is expanded in momentum space around its local equilibrium value in terms of a series of (reducible) Lorentz-tensors formed of particle four-momentum k^μ , i.e., $1, k^\mu, k^\mu k^\nu, \dots$. The procedure adopted by Israel and Stewart is very similar to Grad's: this expansion is truncated at second order in momentum, leaving 14 moments and 14 coefficients in the distribution function to be identified, the so-called 14-moment approximation (in the relativistic case, the bulk viscous pressure does not vanish, leading to one additional moment when compared to Grad's original approach). However, since the expansion is not realized in terms of an orthogonal set, the coefficients of the truncated expansion cannot be immediately determined. For this reason, Israel and Stewart chose a set of constraints to express the expansion coefficients in terms of the main fluid-dynamical variables. Furthermore, since the zeroth and first moments of the Boltzmann equation are the usual conservation laws, it seemed natural to choose the next (the second) moment of the Boltzmann equation to augment and close the conservation equations.

Nevertheless, this choice is ambiguous, since, once the 14-moment approximation is applied, any moment of the Boltzmann equation will lead to a closed set of equations [16,17]. Therefore, inconsistencies may arise because of an ambiguity in the choice of the moment equation for closure. Recently, it was confirmed that, at least for some cases, the IS equations are not in good agreement with the numerical solution of the Boltzmann equation [18,19,20,21,22,23]. Also, the transport coefficients obtained by Israel and Stewart do not coincide with quantum-field theoretical calculations [24].

In this paper, we review the derivation of the fluid-dynamical equations from the Boltzmann equation using the 14-moment approximation, but from a different perspective. First, the single-particle distribution function is expanded around equilibrium in terms of an orthogonal basis. This allows us to determine the coefficients of the expansion without assuming additional constraints. Then, we show how the 14-moment approximation emerges from such a complete moment expansion. Second, we obtain the fluid-dynamical equations for an *arbitrary* moment of the Boltzmann equation and discuss the ambiguity of the 14-moment approximation. We explicitly show that the form of the equations is always the same, regardless of the choice of moment of the Boltzmann equation, but the values of the transport coefficients are different. Thus, we explicitly demonstrate that the traditional 14-moment approximation applied to the Boltzmann equation is not able to provide a unique set of fluid-dynamical equations of motion. In Ref. [17] it is demonstrated how one can resolve this ambiguity by including higher moments of the single-particle distribution function, but this is not the subject of this paper.

The ambiguity in the 14-moment approximation is explicitly demonstrated by calculating the transport coefficients in the 14-moment approximation for a classical gas of massless particles with a constant cross section. We use

two different sets of moments to close the equations of motion: the one used by Israel and Stewart [15], and the one used in Ref. [16], and show how they lead to different transport coefficients. Note that, although our final equations contain terms that were neglected in the papers by Israel and Stewart, see also Refs. [25,26], we will still refer to them as IS equations.

This paper is organized as follows. In Sec. 2 we briefly introduce relativistic fluid dynamics and its dynamical variables. The Boltzmann equation and the definitions of the fluid-dynamical variables from the perspective of kinetic theory are introduced in Sec. 3. The orthonormal basis for the moment expansion and the exact equations for the moments are derived in Sec. 4. In Sec. 5 the 14-moment approximation is applied and the fluid-dynamical equations are derived. The choice of the moment is analysed in Sec. 6. Finally, we conclude in Sec. 7.

Throughout this work we use natural units $\hbar = k_B = c = 1$; the metric tensor is $g^{\mu\nu} = \text{diag}(+, -, -, -)$.

2 Relativistic fluid dynamics

In relativistic fluid dynamics, the variables that specify the macroscopic state of a system are the energy-momentum tensor, $T^{\mu\nu}$, and the particle or net-charge four-current, N^μ . Here we restrict ourselves to only one conserved particle species or net charge. Thus, particle number and energy-momentum conservation imply that

$$\partial_\mu N^\mu = 0, \quad (1)$$

$$\partial_\mu T^{\mu\nu} = 0. \quad (2)$$

In relativistic fluid dynamics, it is useful to define a time-like four-vector, $u^\mu(t, \mathbf{x})$, normalized to $u^\mu u_\mu = 1$, and a projection operator orthogonal to it,

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu, \quad (3)$$

where $\Delta^{\mu\nu} u_\nu \equiv \Delta^{\mu\nu} u_\mu = 0$ and $\Delta^\mu_\mu = 3$. Later on, u^μ will be identified as the fluid four-velocity. From now on, we denote the projection orthogonal to u^μ as $A^{(\mu)} = \Delta^{\mu\nu} A_\nu$, valid for an arbitrary four-vector A^μ . In case of second-rank tensors, $A^{\mu\nu}$, we define the orthogonal and traceless projection as $A^{(\mu\nu)} = \Delta^{\mu\nu\alpha\beta} A_{\alpha\beta}$, where

$$\Delta^{\mu\nu\alpha\beta} = \frac{1}{2} (\Delta^{\mu\alpha} \Delta^{\beta\nu} + \Delta^{\nu\alpha} \Delta^{\beta\mu}) - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\alpha\beta}. \quad (4)$$

Using the projection operator from Eq. (3), the space-time derivative can be decomposed as

$$\partial_\mu = u_\mu D + \nabla_\mu, \quad (5)$$

where the comoving time derivative is $D = u^\mu \partial_\mu$, while the space-like gradient is $\nabla_\mu = \Delta^\nu_\mu \partial_\nu$. For the time derivative we also use the notation $DA = \dot{A}$. Applying the above notation, the relativistic Cauchy-Stokes decomposition is

$$\partial_\mu u_\nu = u_\mu \dot{u}_\nu + \frac{1}{3} \theta \Delta_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \quad (6)$$

where the expansion scalar, θ , the shear tensor, $\sigma^{\mu\nu}$, and the vorticity, $\omega^{\mu\nu}$, are defined as

$$\theta = \nabla_\mu u^\mu, \quad (7)$$

$$\sigma^{\mu\nu} \equiv \nabla^{(\mu} u^{\nu)} = \frac{1}{2} (\nabla^\mu u^\nu + \nabla^\nu u^\mu) - \frac{1}{3} \theta \Delta^{\mu\nu}, \quad (8)$$

$$\omega^{\mu\nu} \equiv \nabla^{[\mu} u^{\nu]} = \frac{1}{2} (\nabla^\mu u^\nu - \nabla^\nu u^\mu). \quad (9)$$

The particle four-current and energy-momentum tensor can be decomposed with respect to u^μ as

$$N^\mu = n u^\mu + V^\mu, \quad (10)$$

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - P \Delta^{\mu\nu} + u^\nu W^\mu + u^\mu W^\nu + \pi^{\mu\nu}, \quad (11)$$

where $n = N^\mu u_\mu$ is the particle density and $\varepsilon = u_\mu T^{\mu\nu} u_\nu$ is the energy density. The trace of the energy-momentum tensor, $P = -\frac{1}{3} \Delta_{\mu\nu} T^{\mu\nu}$, denotes the isotropic pressure. The latter is defined as the sum of the equilibrium pressure P_0 , and the bulk viscous pressure Π , $P = P_0 + \Pi$. The particle diffusion current is defined as

$$V^\mu = \Delta^\mu_\nu N^\nu, \quad (12)$$

while the energy-momentum diffusion current is

$$W^\mu = \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta. \quad (13)$$

The shear-stress tensor, $\pi^{\mu\nu}$, is that part of the energy-momentum tensor that is symmetric, traceless, and orthogonal to u^μ ,

$$\pi^{\mu\nu} = T^{\langle\mu\nu\rangle}. \quad (14)$$

In local thermal equilibrium, the decompositions of Eqs. (10,11) reduce to the ideal-fluid form

$$N_0^\mu = n_0 u^\mu, \quad (15)$$

$$T_0^{\mu\nu} = \varepsilon_0 u^\mu u^\nu - P_0 \Delta^{\mu\nu}. \quad (16)$$

Local thermodynamic equilibrium guarantees that the particle density n_0 , entropy density s_0 , energy density ε_0 , and thermodynamic pressure P_0 , are related to the temperature, T , and chemical potential, μ , through an equation of state (EoS), i.e., $P_0 = P_0(T, \mu)$, from which one can obtain

$$n_0 = \frac{\partial P_0}{\partial \mu}, \quad s_0 = \frac{\partial P_0}{\partial T}, \quad (17)$$

and

$$\varepsilon_0 = T s_0 - P_0 + \mu n_0. \quad (18)$$

In general, the choice of u^μ is ambiguous. The frame where $u^\mu \equiv u_{LR}^\mu = (1, 0, 0, 0)$ is called the local rest frame (LRF) of matter. From the physical perspective, there are two natural choices which fix the LRF but at the same time promote u^μ to a dynamical quantity. According to the definition of Landau and Lifshitz [27], the LRF is tied to the flow of energy-momentum, which leads to

$$u^\mu = \frac{T^{\mu\nu} u_\nu}{\sqrt{T^{\mu\alpha} u_\alpha T_{\mu\beta} u^\beta}}, \quad (19)$$

and thus the energy-momentum diffusion current vanishes, $W^\mu = 0$.

The choice of Eckart [28] relates the LRF to the flow of conserved particles as

$$u^\mu = \frac{N^\mu}{\sqrt{N^\nu N_\nu}}, \quad (20)$$

which implies that the diffusion current vanishes, $V^\mu = 0$. Sometimes it is convenient to introduce the heat flow,

$$q^\mu = W^\mu - h V^\mu, \quad (21)$$

where $h = (\varepsilon + P_0)/n$ is the enthalpy per particle (or per net charge).

Once the four-flow of matter is specified, i.e., replacing the three independent components of W^μ or V^μ by u^μ , we still have to determine 15 independent dynamical variables: six variables, u^μ , ε , n , and P_0 , as in the case of an ideal fluid, and nine variables related to dissipation, Π , q^μ , and $\pi^{\mu\nu}$. Note that the EoS gives one additional constraint and therefore reduces the number of independent variables to 14.

The conservation laws (1,2) constitute only five equations. Thus, to properly close the fluid-dynamical equations it is necessary to introduce nine additional equations which determine the evolution of the remaining dissipative fields, Π , q^μ , and $\pi^{\mu\nu}$. The relativistic extension of Navier-Stokes theory relates the dissipative quantities to gradients of the primary fluid-dynamical fields,

$$\Pi = -\zeta \theta, \quad (22)$$

$$q^\mu = -\kappa_q \frac{T^2}{h} \nabla^\mu \left(\frac{\mu}{T} \right), \quad (23)$$

$$\pi^{\mu\nu} = 2\eta \sigma^{\mu\nu}, \quad (24)$$

where the bulk viscosity coefficient ζ , the heat-flow coefficient κ_q , and the shear viscosity coefficient η are positive-definite functions of T and μ .

However, as mentioned in the introduction this naive approach leads to intrinsic problems, such as acausal signal propagation and instabilities, and is therefore unsuitable to describe relativistic fluids. The acausality problems were solved by introducing memory effects into the definitions of Π , q^μ , and $\pi^{\mu\nu}$, which are no longer assumed to be linearly related to gradients of the primary fluid-dynamical variables [29,30,31,32,33,34,35]. Instead, they become independent dynamical variables that obey dynamical equations of motion (which introduce the relaxation times τ_π , τ_Π , and τ_q) that describe their transient dynamics towards their respective asymptotic relativistic Navier-Stokes solution,

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta \theta + \dots, \quad (25)$$

$$\tau_q \dot{q}^{\langle\mu} + q^\mu = -\kappa_q \frac{T^2}{h} \nabla^\mu \left(\frac{\mu}{T} \right) + \dots, \quad (26)$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu} + \pi^{\mu\nu} = 2\eta \sigma^{\mu\nu} + \dots, \quad (27)$$

where $\dot{q}^{\langle\mu} = \Delta^\mu_\alpha D q^\alpha$ and $\dot{\pi}^{\langle\mu\nu} = \Delta^\mu_\alpha \Delta^\nu_\beta D \pi^{\alpha\beta}$ and the dots denote possible higher-order terms. These are the type

of equations of motion which can also be derived from relativistic kinetic theory as shown by Israel and Stewart and others [8, 15, 26, 36]. Causality is guaranteed, provided the relaxation times fulfill certain constraints [4].

3 The relativistic Boltzmann equation

Let us consider a relativistic dilute gas characterized only by the single-particle distribution function $f_{\mathbf{k}} \equiv f(x^\mu, k^\mu)$, the evolution of which is given by the relativistic Boltzmann equation [7],

$$k^\mu \partial_\mu f_{\mathbf{k}} = C[f], \quad (28)$$

where $k^\mu = (k^0, \mathbf{k})$ with $k^0 = \sqrt{\mathbf{k}^2 + m^2}$ and m being the mass of the particles. For the collision term $C[f]$, we consider only elastic two-to-two collisions with incoming momenta k, k' , and outgoing momenta p, p' ,

$$C[f] = \frac{1}{\nu} \int dK' dP dP' W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'} \times (f_{\mathbf{p}} f_{\mathbf{p}'} \tilde{f}_{\mathbf{k}} \tilde{f}_{\mathbf{k}'} - f_{\mathbf{k}} f_{\mathbf{k}'} \tilde{f}_{\mathbf{p}} \tilde{f}_{\mathbf{p}'}), \quad (29)$$

where $\nu = 2$ is a symmetry factor. The Lorentz-invariant phase volume is $dK \equiv g d^3\mathbf{k} / [(2\pi)^3 k^0]$, with g being the number of internal degrees of freedom. The Lorentz-invariant transition rate $W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'}$ is symmetric with respect to the exchange of the outgoing momenta, as well as to time reversal,

$$W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'} \equiv W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}'\mathbf{p}} = W_{\mathbf{p}\mathbf{p}' \rightarrow \mathbf{k}\mathbf{k}'}. \quad (30)$$

Here, we also take into account quantum statistics and introduced the notation $\tilde{f}_{\mathbf{k}} \equiv 1 - a f_{\mathbf{k}}$, where $a = 1$ ($a = -1$) for fermions (bosons) and $a = 0$ in the limiting case of classical Boltzmann-Gibbs statistics.

The particle four-flow and the energy-momentum tensor are identified as the first and second moments of the single-particle distribution function,

$$N^\mu = \langle k^\mu \rangle, \quad (31)$$

$$T^{\mu\nu} = \langle k^\mu k^\nu \rangle, \quad (32)$$

where we adopted the following notation for the averages

$$\langle \dots \rangle = \int dK (\dots) f_{\mathbf{k}}. \quad (33)$$

Making use of the properties of the transition rate (30), one can show [7] that the particle four-flow and the energy-momentum tensor satisfy the conservation equations (1,2) for any solution of the Boltzmann equation,

$$\partial_\mu \langle k^\mu \rangle \equiv \int dK C[f] = 0, \quad (34)$$

$$\partial_\mu \langle k^\mu k^\nu \rangle \equiv \int dK k^\nu C[f] = 0. \quad (35)$$

In order to identify the macroscopic variables introduced in Eqs. (10, 11) in terms of the single-particle distribution function we decompose the momentum of the

particles k^μ into two parts: one parallel to the flow velocity u^μ and the other orthogonal to the latter,

$$k^\mu = E_{\mathbf{k}} u^\mu + k^{\langle\mu\rangle}, \quad (36)$$

where we defined the energy of a particle as $E_{\mathbf{k}} \equiv u_\mu k^\mu$. Using the above decomposition in Eqs. (31,32) we obtain

$$N^\mu = \langle E_{\mathbf{k}} \rangle u^\mu + \langle k^{\langle\mu\rangle} \rangle, \quad (37)$$

$$T^{\mu\nu} = \langle E_{\mathbf{k}}^2 \rangle u^\mu u^\nu + \frac{1}{3} \Delta^{\mu\nu} \langle \Delta^{\alpha\beta} k_\alpha k_\beta \rangle + u^\nu \langle E_{\mathbf{k}} k^{\langle\mu\rangle} \rangle + u^\mu \langle E_{\mathbf{k}} k^{\langle\nu\rangle} \rangle + \langle k^{\langle\mu} k^{\nu\rangle} \rangle. \quad (38)$$

Comparing these decompositions with Eqs. (10,11), we identify the main fluid-dynamical quantities as averages or moments with respect to an arbitrary solution of the Boltzmann equation,

$$n = \langle E_{\mathbf{k}} \rangle, \quad \varepsilon = \langle E_{\mathbf{k}}^2 \rangle, \quad P_0 + \Pi = -\frac{1}{3} \langle \Delta^{\mu\nu} k_\mu k_\nu \rangle, \quad V^\mu = \langle k^{\langle\mu\rangle} \rangle, \quad W^\mu = \langle E_{\mathbf{k}} k^{\langle\mu\rangle} \rangle, \quad \pi^{\mu\nu} = \langle k^{\langle\mu} k^{\nu\rangle} \rangle. \quad (39)$$

Similarly, we introduce the average with respect to the local equilibrium distribution function $f_{0\mathbf{k}}$,

$$\langle \dots \rangle_0 = \int dK (\dots) f_{0\mathbf{k}}, \quad (40)$$

where

$$f_{0\mathbf{k}}(x^\mu, k^\mu) = [\exp(\beta_0 E_{\mathbf{k}} - \alpha_0) + a]^{-1}. \quad (41)$$

Although $f_{0\mathbf{k}}$ satisfies detailed balance, it is not a solution of the Boltzmann equation. The quantities $\alpha_0(x^\mu)$ and $\beta_0(x^\mu)$ are defined for an arbitrary non-equilibrium distribution function $f_{\mathbf{k}}$ by the matching conditions,

$$n \equiv n_0 = \langle E_{\mathbf{k}} \rangle_0, \quad \varepsilon \equiv \varepsilon_0 = \langle E_{\mathbf{k}}^2 \rangle_0. \quad (42)$$

In local equilibrium we would then identify $\beta_0 = 1/T$ as the inverse temperature and $\alpha_0 = \mu/T$ as the ratio of chemical potential over temperature. The matching conditions (42) lead to

$$\langle E_{\mathbf{k}} \rangle_\delta = 0, \quad \langle E_{\mathbf{k}}^2 \rangle_\delta = 0, \quad (43)$$

where $\langle \dots \rangle_\delta \equiv \langle \dots \rangle - \langle \dots \rangle_0$. The matching conditions (42) are convenient as they allow us to use equilibrium thermodynamic relations between n, ε, P_0, T , and μ . We also note that the EoS is not an additional input, but follows from the single-particle distribution function in local equilibrium.

Finally, the separation between thermodynamic pressure and bulk viscous pressure is achieved as

$$P_0 = -\frac{1}{3} \langle \Delta^{\mu\nu} k_\mu k_\nu \rangle_0, \quad \Pi = -\frac{1}{3} \langle \Delta^{\mu\nu} k_\mu k_\nu \rangle_\delta. \quad (44)$$

We remark that $\langle k^{\langle\mu\rangle} \rangle_0 = \langle E_{\mathbf{k}} k^{\langle\mu\rangle} \rangle_0 = \langle k^{\langle\mu} k^{\nu\rangle} \rangle_0 = 0$ guarantees that in local thermal equilibrium all dissipative currents vanish.

4 Moment expansion

The method of moments is the most common approach to derive the so-called second-order theories from kinetic theory. In this section, we review the basic ideas of this method along the lines of Refs. [7,8]. Since we are interested in near-equilibrium solutions of the Boltzmann equation, we start by expanding $f_{\mathbf{k}}$ around the local equilibrium distribution function $f_{0\mathbf{k}}$,

$$f_{\mathbf{k}} \equiv f_{0\mathbf{k}} + \delta f_{\mathbf{k}} = f_{0\mathbf{k}} \left(1 + \tilde{f}_{0\mathbf{k}} \phi_{\mathbf{k}} \right), \quad (45)$$

where $\phi_{\mathbf{k}}$ represents a general non-equilibrium correction.

Following Ref. [7], $\phi_{\mathbf{k}}$ is expanded in momentum space with the help of the irreducible tensors 1 , $k^{(\mu)}$, $k^{(\mu} k^{\nu)}$, $k^{(\mu} k^{\nu} k^{\lambda)}$, \dots , forming a complete and orthogonal set, analogous to the spherical harmonics [14]. These irreducible tensors are defined by using the symmetrized traceless projections as

$$k^{(\mu_1} \dots k^{\mu_m)} = \Delta^{\mu_1 \dots \mu_m \nu_1 \dots \nu_m} k_{\nu_1} \dots k_{\nu_m}, \quad (46)$$

see also Ref. [37]. The tensors $k^{(\mu_1} \dots k^{\mu_m)}$ satisfy the following orthogonality condition,

$$\int dK F_{\mathbf{k}} k^{(\mu_1} \dots k^{\mu_m)} k^{(\nu_1} \dots k^{\nu_n)} = \frac{m! \delta_{mn}}{(2m+1)!!} \Delta^{\mu_1 \dots \mu_m \nu_1 \dots \nu_m} \int dK F_{\mathbf{k}} (\Delta^{\alpha\beta} k_{\alpha} k_{\beta})^m. \quad (47)$$

Here $m, n = 0, 1, 2, \dots$, $F_{\mathbf{k}}$ is an arbitrary scalar function of $E_{\mathbf{k}}$, and $(2m+1)!!$ denotes the double factorial.

Using these tensors as the basis of the expansion, the non-equilibrium correction can be written as,

$$\phi_{\mathbf{k}} = \sum_{\ell=0}^{\infty} \lambda_{\mathbf{k}}^{(\mu_1 \dots \mu_{\ell})} k^{(\mu_1} \dots k^{\mu_{\ell})}, \quad (48)$$

where the index ℓ indicates the rank of the tensor $\lambda_{\mathbf{k}}^{(\mu_1 \dots \mu_{\ell})}$, and $\ell = 0$ corresponds to the scalar λ . The coefficients $\lambda_{\mathbf{k}}^{(\mu_1 \dots \mu_{\ell})}$ may be further expanded in energy $E_{\mathbf{k}}$ with another orthogonal basis of functions $P_{\mathbf{k}n}^{(\ell)}$,

$$\lambda_{\mathbf{k}}^{(\mu_1 \dots \mu_{\ell})} = \sum_{n=0}^{N_{\ell}} c_n^{(\mu_1 \dots \mu_{\ell})} P_{\mathbf{k}n}^{(\ell)}, \quad (49)$$

where $c_n^{(\mu_1 \dots \mu_{\ell})}$ are as of yet undetermined coefficients and N_{ℓ} is the number of functions $P_{\mathbf{k}n}^{(\ell)}$ considered to describe the ℓ -th rank tensor $\lambda_{\mathbf{k}}^{(\mu_1 \dots \mu_{\ell})}$. The functions $P_{\mathbf{k}n}^{(\ell)}$ are chosen to be polynomials of order n in energy,

$$P_{\mathbf{k}n}^{(\ell)} = \sum_{r=0}^n a_{nr}^{(\ell)} E_{\mathbf{k}}^r, \quad (50)$$

and are constructed using the following orthonormality condition,

$$\int dK \omega^{(\ell)} P_{\mathbf{k}m}^{(\ell)} P_{\mathbf{k}n}^{(\ell)} = \delta_{mn}, \quad (51)$$

where the weight $\omega^{(\ell)}$ is defined as

$$\omega^{(\ell)} = \frac{\mathcal{N}^{(\ell)}}{(2\ell+1)!!} (\Delta^{\alpha\beta} k_{\alpha} k_{\beta})^{\ell} f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}. \quad (52)$$

The coefficients $a_{nr}^{(\ell)}$ and the normalization constants $\mathcal{N}^{(\ell)}$ can be found via Gram-Schmidt orthogonalization using the orthonormality condition (51), see Sec. 5 or Ref. [17] for more details.

Finally, the single-particle distribution function can be expressed as

$$f_{\mathbf{k}} = f_{0\mathbf{k}} \left(1 + \tilde{f}_{0\mathbf{k}} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_{\ell}} \mathcal{H}_{\mathbf{k}n}^{(\ell)} \rho_n^{\mu_1 \dots \mu_{\ell}} k^{(\mu_1} \dots k^{\mu_{\ell})} \right), \quad (53)$$

where we introduced the energy-dependent coefficients

$$\mathcal{H}_{\mathbf{k}n}^{(\ell)} = \frac{\mathcal{N}^{(\ell)}}{\ell!} \sum_{m=n}^{N_{\ell}} a_{mn}^{(\ell)} P_{\mathbf{k}m}^{(\ell)} \quad (54)$$

and the generalized irreducible moment of $\delta f_{\mathbf{k}}$,

$$\rho_r^{\mu_1 \dots \mu_{\ell}} = \left\langle E_{\mathbf{k}}^r k^{(\mu_1} \dots k^{\mu_{\ell})} \right\rangle_{\delta}, \quad (55)$$

with

$$\langle \dots \rangle_{\delta} = \int dK (\dots) \delta f_{\mathbf{k}}. \quad (56)$$

Using this notation, the expansion coefficients in Eq. (49) can be immediately determined using Eqs. (47) and (51). For $n \leq N_{\ell}$ they are given by

$$\begin{aligned} c_n^{(\mu_1 \dots \mu_{\ell})} &\equiv \frac{\mathcal{N}^{(\ell)}}{\ell!} \left\langle P_{\mathbf{k}n}^{(\ell)} k^{(\mu_1} \dots k^{\mu_{\ell})} \right\rangle_{\delta} \\ &= \frac{\mathcal{N}^{(\ell)}}{\ell!} \sum_{r=0}^n \rho_r^{\mu_1 \dots \mu_{\ell}} a_{nr}^{(\ell)}. \end{aligned} \quad (57)$$

Naturally, the dissipative currents are related to the tensors $\rho_r^{\mu_1 \dots \mu_{\ell}}$. According to Eqs. (39) we can identify them as

$$\rho_0 = -3\Pi/m^2, \quad (58)$$

$$\rho_0^{\mu} = V^{\mu}, \quad (59)$$

$$\rho_1^{\mu} = W^{\mu}, \quad (60)$$

$$\rho_0^{\mu\nu} = \pi^{\mu\nu}. \quad (61)$$

Furthermore, the matching conditions imposed in Eq. (43) can also be recast using the irreducible moments,

$$\rho_1 = \rho_2 = 0. \quad (62)$$

The definition of the LRF corresponding to Landau's choice (19) requires that

$$\rho_1^{\mu} = 0, \quad (63)$$

while Eckart's definition (20) leads to

$$\rho_0^{\mu} = 0. \quad (64)$$

4.1 General equations of motion

So far, the single-particle distribution function was expressed in terms of the irreducible tensors $\rho_n^{\mu_1 \dots \mu_\ell}$. The time-evolution equations for these tensors can be obtained directly from the Boltzmann equation by applying the comoving derivative to Eq. (55) together with the symmetrized traceless projection,

$$\dot{\rho}_r^{\langle \mu_1 \dots \mu_\ell \rangle} = \Delta_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell} \frac{d}{d\tau} \int dK E_{\mathbf{k}}^r k^{\langle \nu_1} \dots k^{\nu_\ell \rangle} \delta f_{\mathbf{k}}. \quad (65)$$

Now, using the Boltzmann equation (28) in the form

$$\delta \dot{f}_{\mathbf{k}} = -\dot{f}_{0\mathbf{k}} - E_{\mathbf{k}}^{-1} k_\nu \nabla^\nu f_{0\mathbf{k}} - E_{\mathbf{k}}^{-1} k_\nu \nabla^\nu \delta f_{\mathbf{k}} + E_{\mathbf{k}}^{-1} C[f], \quad (66)$$

and substituting into Eq. (65), we obtain the *exact* equations for $\rho_r^{\mu_1 \dots \mu_\ell}$.

Since fluid dynamics does not involve tensors of rank higher than two, it is sufficient to derive the time-evolution equations for the fields ρ_r , ρ_r^μ , and $\rho_r^{\mu\nu}$ only. Similar equations could also be derived for the higher-rank irreducible tensors, if needed. Thus, using Eqs. (65) and (66), the equation for an arbitrary scalar moment is

$$\begin{aligned} \dot{\rho}_r &= C_{r-1} + \alpha_r^{(0)} \theta + \left(r \rho_{r-1}^\mu + \frac{G_{2r}}{D_{20}} W^\mu \right) \dot{u}_\mu \\ &\quad - \nabla_\mu \rho_{r-1}^\mu + \frac{G_{3r}}{D_{20}} \partial_\mu V^\mu - \frac{G_{2r}}{D_{20}} \partial_\mu W^\mu \\ &\quad + \frac{1}{3} \left[(r-1) m^2 \rho_{r-2} - (r+2) \rho_r - 3 \frac{G_{2r}}{D_{20}} \Pi \right] \theta \\ &\quad + \left[(r-1) \rho_{r-2}^{\mu\nu} + \frac{G_{2r}}{D_{20}} \pi^{\mu\nu} \right] \sigma_{\mu\nu}. \end{aligned} \quad (67)$$

Similarly, the time-evolution equation for the vector moment is

$$\begin{aligned} \dot{\rho}_r^{\langle \mu \rangle} &= C_{r-1}^{\langle \mu \rangle} + \alpha_r^{(1)} \nabla^\mu \alpha_0 - \alpha_r^h \dot{W}^\mu + r \rho_{r-1}^{\mu\nu} \dot{u}_\nu \\ &\quad + \frac{1}{3} \left[r m^2 \rho_{r-1}^\mu - (r+3) \rho_{r+1}^\mu - 3 \alpha_r^h \Pi \right] \dot{u}^\mu \\ &\quad - \frac{1}{3} \nabla^\mu \left(m^2 \rho_{r-1}^\mu - \rho_{r+1}^\mu \right) + \alpha_r^h \nabla^\mu \Pi \\ &\quad - \Delta_\nu^\mu \left(\nabla_\lambda \rho_{r-1}^{\nu\lambda} + \alpha_r^h \partial_\lambda \pi^{\nu\lambda} \right) \\ &\quad + \frac{1}{3} \left[(r-1) m^2 \rho_{r-2}^\mu - (r+3) \rho_r^\mu - 4 \alpha_r^h W^\mu \right] \theta \\ &\quad + \frac{1}{5} \left[(2r-2) m^2 \rho_{r-2}^\nu - (2r+3) \rho_r^\nu - 5 \alpha_r^h W^\nu \right] \sigma_\nu^\mu \\ &\quad + \left(\rho_r^\nu + \alpha_r^h W^\nu \right) \omega_\nu^\mu + (r-1) \rho_{r-2}^{\mu\nu\lambda} \sigma_{\nu\lambda}, \end{aligned} \quad (68)$$

while the equation for $\rho_r^{\mu\nu}$ is

$$\begin{aligned} \dot{\rho}_r^{\langle \mu\nu \rangle} &= C_{r-1}^{\langle \mu\nu \rangle} + 2 \alpha_r^{(2)} \sigma^{\mu\nu} + \frac{2}{15} \left[(r-1) m^4 \rho_{r-2} \right. \\ &\quad \left. - (2r+3) m^2 \rho_r + (r+4) \rho_{r+2} \right] \sigma^{\mu\nu} \\ &\quad + \frac{2}{5} \left[r m^2 \rho_{r-1}^{\langle \mu} - (r+5) \rho_{r+1}^{\langle \mu} \right] \dot{u}^{\nu \rangle} \\ &\quad + r \rho_{r-1}^{\mu\nu\lambda} \dot{u}_\lambda - \frac{2}{5} \nabla^{\langle \mu} \left(m^2 \rho_{r-1}^{\nu \rangle} - \rho_{r+1}^{\nu \rangle} \right) \\ &\quad + \frac{1}{3} \left[(r-1) m^2 \rho_{r-2}^{\mu\nu} - (r+4) \rho_r^{\mu\nu} \right] \theta \\ &\quad + \frac{2}{7} \left[(2r-2) m^2 \rho_{r-2}^{\lambda \langle \mu} - (2r+5) \rho_r^{\lambda \langle \mu} \right] \sigma_\lambda^{\nu \rangle} \\ &\quad + 2 \rho_r^{\lambda \langle \mu} \omega_\lambda^{\nu \rangle} - \Delta_{\alpha\beta}^{\mu\nu} \nabla_\lambda \rho_{r-1}^{\alpha\beta\lambda} + (r-1) \rho_{r-2}^{\mu\nu\lambda\kappa} \sigma_{\lambda\kappa}. \end{aligned} \quad (69)$$

Here we introduced the generalized collision term

$$\begin{aligned} C_r^{\langle \mu_1 \dots \mu_\ell \rangle} &\equiv \Delta_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell} C_{r-1}^{\nu_1 \dots \nu_\ell} \\ &= \Delta_{\nu_1 \dots \nu_\ell}^{\mu_1 \dots \mu_\ell} \int dK E_{\mathbf{k}}^r k^{\mu_1} \dots k^{\mu_\ell} C[f]. \end{aligned} \quad (70)$$

All derivatives of α_0 and β_0 that appear in the above equations were replaced using the following equations, obtained from the conservation laws (1) and (2),

$$\begin{aligned} \dot{\alpha}_0 &= \frac{1}{D_{20}} \left[-J_{30} (n_0 \theta + \partial_\mu V^\mu) + J_{20} (\varepsilon_0 + P_0 + \Pi) \theta \right. \\ &\quad \left. + J_{20} (\partial_\mu W^\mu - W^\mu \dot{u}_\mu - \pi^{\mu\nu} \sigma_{\mu\nu}) \right], \end{aligned} \quad (71)$$

$$\begin{aligned} \dot{\beta}_0 &= \frac{1}{D_{20}} \left[-J_{20} (n_0 \theta + \partial_\mu V^\mu) + J_{10} (\varepsilon_0 + P_0 + \Pi) \theta \right. \\ &\quad \left. + J_{10} (\partial_\mu W^\mu - W^\mu \dot{u}_\mu - \pi^{\mu\nu} \sigma_{\mu\nu}) \right], \end{aligned} \quad (72)$$

$$\begin{aligned} \dot{u}^\mu &= \beta_0^{-1} \left(h_0^{-1} \nabla^\mu \alpha_0 - \nabla^\mu \beta_0 \right) - \frac{h_0^{-1}}{n_0} (\Pi \dot{u}^\mu - \nabla^\mu \Pi) \\ &\quad - \frac{h_0^{-1}}{n_0} \left[\frac{4}{3} W^\mu \theta + W_\nu (\sigma^{\mu\nu} - \omega^{\mu\nu}) + \dot{W}^\mu + \Delta_\nu^\mu \partial_\lambda \pi^{\nu\lambda} \right], \end{aligned} \quad (73)$$

where $h_0 = (\varepsilon_0 + P_0)/n_0$. The coefficients α_r are functions of thermodynamic variables,

$$\alpha_r^{(0)} = (1-r) I_{r1} - I_{r0} - \frac{n_0}{D_{20}} (h_0 G_{2r} - G_{3r}), \quad (74)$$

$$\alpha_r^{(1)} = J_{r+1,1} - h_0^{-1} J_{r+2,1}, \quad (75)$$

$$\alpha_r^{(2)} = I_{r+2,1} + (r-1) I_{r+2,2}, \quad (76)$$

$$\alpha_r^h = -\frac{\beta_0}{\varepsilon_0 + P_0} J_{r+2,1}, \quad (77)$$

where we used the notation

$$I_{nq} = \frac{(-1)^q}{(2q+1)!!} \int dK E_{\mathbf{k}}^{n-2q} (\Delta^{\alpha\beta} k_\alpha k_\beta)^q f_{0\mathbf{k}}, \quad (78)$$

$$J_{nq} = \frac{(-1)^q}{(2q+1)!!} \int dK E_{\mathbf{k}}^{n-2q} (\Delta^{\alpha\beta} k_\alpha k_\beta)^q f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}, \quad (79)$$

$$G_{nm} = J_{n0} J_{m0} - J_{n-1,0} J_{m+1,0}, \quad (80)$$

$$D_{nq} = J_{n+1,q} J_{n-1,q} - (J_{nq})^2. \quad (81)$$

Thus, we have obtained an infinite set of coupled equations containing all moments of the distribution function. Note that the derivation of these equations is independent of the form of the expansion we introduced in the previous subsection.

5 The 14-moment approximation

In order to obtain the macroscopic equations of motion in terms of the fluid-dynamical variables that appear in the particle four-current and energy-momentum tensor, the generic hierarchy of the coupled moment equations must be truncated. To this end, Israel and Stewart made the so-called 14-moment approximation: they truncated the expansion of the distribution function and matched the non-equilibrium corrections to the dissipative currents Π , V^μ , W^μ , and $\pi^{\mu\nu}$.

In this section, we show how the 14-moment approximation emerges from the general moment expansion presented in Sec. 4. First, we neglect irreducible tensor moments of rank higher than two, i.e., $\rho_r^{\mu_1 \dots \mu_\ell} = 0$ for $\ell \geq 3$ in Eqs. (68,69). Such irreducible moments cannot be constructed purely from first-order gradients of equilibrium fields [17]. This means that they lead to terms that are of higher order in gradients or contain higher powers of dissipative quantities in the equations of motion.

Next, in the expansion (53) of the distribution function we include the first three scalar moments, namely $\rho_0 = -3\Pi/m^2$, $\rho_1 = 0$, and $\rho_2 = 0$ (the last two scalar moments vanish due to the matching condition, but must be included since they were used to define α_0 and β_0), the first two vector moments, $\rho_0^\mu = V^\mu$ and $\rho_1^\mu = W^\mu$, and the first second-rank tensor moment $\rho_0^{\mu\nu} = \pi^{\mu\nu}$. This implies that $N_0 = 2$, $N_1 = 1$, and $N_2 = 0$, while all other moments appearing in the expansion are dropped. Choosing either the Eckart or the Landau frame, we can eliminate either V^μ or W^μ , respectively. Let us note that, so far, this approach is completely equivalent to the matching procedure of Israel and Stewart. The ambiguity in the transport coefficients emerges only at a later stage, when choosing moments of the Boltzmann equation to supply the equations of motion for the dissipative currents.

The restriction to the aforementioned moments affects that $\lambda_{\mathbf{k}}$, $\lambda_{\mathbf{k}}^{\langle\mu\rangle}$, and $\lambda_{\mathbf{k}}^{\langle\mu\nu\rangle}$ from Eq. (49) can be expressed solely in terms of Π , V^μ , W^μ , and $\pi^{\mu\nu}$,

$$\lambda_{\mathbf{k}} \equiv \sum_{n=0}^{N_0} c_n P_{\mathbf{k}n}^{(0)} \simeq c_0 P_{\mathbf{k}0}^{(0)} + c_1 P_{\mathbf{k}1}^{(0)} + c_2 P_{\mathbf{k}2}^{(0)}, \quad (82)$$

$$\lambda_{\mathbf{k}}^{\langle\mu\rangle} \equiv \sum_{n=0}^{N_1} c_n^{\langle\mu\rangle} P_{\mathbf{k}n}^{(1)} \simeq c_0^{\langle\mu\rangle} P_{\mathbf{k}0}^{(1)} + c_1^{\langle\mu\rangle} P_{\mathbf{k}1}^{(1)}, \quad (83)$$

$$\lambda_{\mathbf{k}}^{\langle\mu\nu\rangle} \equiv \sum_{n=0}^{N_2} c_n^{\langle\mu\nu\rangle} P_{\mathbf{k}n}^{(2)} \simeq c_0^{\langle\mu\nu\rangle} P_{\mathbf{k}0}^{(2)}, \quad (84)$$

where the tensors $c_n^{\langle\mu_1 \dots \mu_\ell\rangle}$ are given by Eq. (57), while those which do not appear in the above equations are set

to zero. According to Eq. (57) the scalars c_0 , c_1 , and c_2 are proportional to the bulk viscous pressure,

$$c_0 = -\frac{3\Pi}{m^2} a_{00}^{(0)} \mathcal{N}^{(0)}, \quad (85)$$

$$c_1 = -\frac{3\Pi}{m^2} a_{10}^{(0)} \mathcal{N}^{(0)}, \quad (86)$$

$$c_2 = -\frac{3\Pi}{m^2} a_{20}^{(0)} \mathcal{N}^{(0)}. \quad (87)$$

The vectors $c_0^{\langle\mu\rangle}$ and $c_1^{\langle\mu\rangle}$ are given by a linear combination of particle and energy-momentum diffusion currents,

$$c_0^{\langle\mu\rangle} = V^\mu a_{00}^{(1)} \mathcal{N}^{(1)}, \quad (88)$$

$$c_1^{\langle\mu\rangle} = V^\mu a_{10}^{(1)} \mathcal{N}^{(1)} + W^\mu a_{11}^{(1)} \mathcal{N}^{(1)}, \quad (89)$$

while $c_0^{\langle\mu\nu\rangle}$ is proportional to the shear-stress tensor,

$$c_0^{\langle\mu\nu\rangle} = \pi^{\mu\nu} a_{00}^{(2)} \frac{\mathcal{N}^{(2)}}{2}. \quad (90)$$

Let us recall Eq. (50) and for any $\ell \geq 0$ we set

$$P_{\mathbf{k}0}^{(\ell)} \equiv a_{00}^{(\ell)} = 1, \quad (91)$$

while

$$P_{\mathbf{k}1}^{(0)} = a_{11}^{(0)} E_{\mathbf{k}} + a_{10}^{(0)}, \quad (92)$$

$$P_{\mathbf{k}1}^{(1)} = a_{11}^{(1)} E_{\mathbf{k}} + a_{10}^{(1)}, \quad (93)$$

$$P_{\mathbf{k}2}^{(0)} = a_{22}^{(0)} E_{\mathbf{k}}^2 + a_{21}^{(0)} E_{\mathbf{k}} + a_{20}^{(0)}. \quad (94)$$

The orthonormality condition (51) implies that the normalization constant is

$$\mathcal{N}^{(\ell)} = (J_{2\ell,\ell})^{-1}, \quad (95)$$

and

$$\frac{a_{10}^{(0)}}{a_{11}^{(0)}} = -\frac{J_{10}}{J_{00}}, \quad (a_{11}^{(0)})^2 = \frac{J_{00}^2}{D_{10}}, \quad (96)$$

$$\frac{a_{21}^{(0)}}{a_{22}^{(0)}} = \frac{G_{12}}{D_{10}}, \quad \frac{a_{20}^{(0)}}{a_{22}^{(0)}} = \frac{D_{20}}{D_{10}}, \quad (97)$$

$$(a_{22}^{(0)})^2 = \frac{J_{00} D_{10}}{J_{20} D_{20} + J_{30} G_{12} + J_{40} D_{10}}, \quad (98)$$

$$\frac{a_{10}^{(1)}}{a_{11}^{(1)}} = -\frac{J_{31}}{J_{21}}, \quad (a_{11}^{(1)})^2 = \frac{J_{21}^2}{D_{31}}. \quad (99)$$

Furthermore, using the orthogonality relation (47) together with Eqs. (48-50) one can easily show that

$$\rho_r^{\mu_1 \dots \mu_\ell} = \ell! \sum_{n=0}^{N_\ell} \sum_{m=0}^n c_n^{\langle\mu_1 \dots \mu_\ell\rangle} a_{nm}^{(\ell)} J_{r+m+2\ell,\ell}. \quad (100)$$

Applying the truncation scheme required by the 14-moment approximation we obtain that all scalar moments, ρ_r , become proportional to the bulk viscous pressure Π ,

$$\rho_r \equiv \sum_{n=0}^{N_0} \sum_{m=0}^n c_n a_{nm}^{(0)} J_{r+m,0} = \gamma_r^\Pi \Pi. \quad (101)$$

Similarly, all vector moments, ρ_r^μ , are proportional to a linear combination of V^μ and W^μ ,

$$\rho_r^\mu \equiv \sum_{n=0}^{N_1} \sum_{m=0}^n c_n^{(\mu)} a_{nm}^{(1)} J_{r+m+2,1} = \gamma_r^V V^\mu + \gamma_r^W W^\mu, \quad (102)$$

and, finally, $\rho_r^{\mu\nu}$ is proportional to $\pi^{\mu\nu}$,

$$\rho_r^{\mu\nu} \equiv \sum_{n=0}^{N_2} \sum_{m=0}^n c_n^{(\mu\nu)} a_{nm}^{(2)} J_{r+m+4,2} = \gamma_r^\pi \pi^{\mu\nu}. \quad (103)$$

Now, using the previously obtained results we prove that

$$\gamma_r^\Pi = \mathcal{A}_\Pi J_{r0} + \mathcal{B}_\Pi J_{r+1,0} + \mathcal{C}_\Pi J_{r+2,0}, \quad (104)$$

$$\gamma_r^V = \mathcal{A}_V J_{r+2,1} + \mathcal{B}_V J_{r+3,1}, \quad (105)$$

$$\gamma_r^W = \mathcal{A}_W J_{r+2,1} + \mathcal{B}_W J_{r+3,1}, \quad (106)$$

$$\gamma_r^\pi = 2\mathcal{A}_\pi J_{r+4,2}, \quad (107)$$

where

$$\mathcal{A}_\Pi = -\frac{3}{m^2} \frac{D_{30}}{J_{20}D_{20} + J_{30}G_{12} + J_{40}D_{10}}, \quad (108)$$

$$\mathcal{B}_\Pi = -\frac{3}{m^2} \frac{G_{23}}{J_{20}D_{20} + J_{30}G_{12} + J_{40}D_{10}}, \quad (109)$$

$$\mathcal{C}_\Pi = -\frac{3}{m^2} \frac{D_{20}}{J_{20}D_{20} + J_{30}G_{12} + J_{40}D_{10}}, \quad (110)$$

$$\mathcal{A}_V = \frac{J_{41}}{D_{31}}, \quad \mathcal{A}_W = -\frac{J_{31}}{D_{31}}, \quad (111)$$

$$\mathcal{B}_V = -\frac{J_{31}}{D_{31}}, \quad \mathcal{B}_W = \frac{J_{21}}{D_{31}}, \quad (112)$$

$$\mathcal{A}_\pi = \frac{1}{2J_{42}}. \quad (113)$$

We remark that, since the matching conditions were already imposed, one can prove that $\gamma_1^\Pi = \gamma_2^\Pi = 0$.

Equations (101), (102), and (103) are the main result of the 14-moment approximation. Such relations guarantee that any irreducible moment of the distribution function can be expressed in terms of the dissipative currents appearing in N^μ and $T^{\mu\nu}$. This is also what Israel and Stewart achieved by their matching procedure. Consequently, a closed set of fluid-dynamical equations can always be derived. This happens because the reduction of dynamical variables was done directly in the single-particle distribution. On the other hand, this truncation also leads to an ambiguity in the derivation of fluid-dynamical equations since, for example, the equation of motion for the bulk viscous pressure can be obtained from ρ_r for any r . We will come back to this point in Sec. 6.

5.1 The collision term

In order to express the collision term (70) in terms of the fundamental fluid variables, $C[f]$ is linearized in deviations from the equilibrium distribution function. Substituting Eq. (45) into the linearized collision term, we

obtain

$$C_{r-1}^{\mu_1 \dots \mu_\ell} = \frac{1}{\nu} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} \times E_{\mathbf{k}}^{r-1} k^{\mu_1} \dots k^{\mu_\ell} (\phi_{\mathbf{p}} + \phi_{\mathbf{p}'} - \phi_{\mathbf{k}} - \phi_{\mathbf{k}'}), \quad (114)$$

where the ϕ 's are given in Eq. (48). In order to specify C_{r-1} , $C_{r-1}^{(\alpha)}$, and $C_{r-1}^{(\alpha\beta)}$ for the general equations of motion (67, 68, 69) we project the collision term as

$$C_{r-1} = u_{\mu_1} \dots u_{\mu_\ell} C_{r-\ell-1}^{\mu_1 \dots \mu_\ell}, \quad (115)$$

$$C_{r-1}^{(\alpha)} = \Delta_{\mu_1}^\alpha u_{\mu_2} \dots u_{\mu_\ell} C_{r-\ell}^{\mu_1 \dots \mu_\ell}, \quad (116)$$

$$C_{r-1}^{(\alpha\beta)} = \Delta_{\mu_1 \mu_2}^{\alpha\beta} u_{\mu_3} \dots u_{\mu_\ell} C_{r-\ell+1}^{\mu_1 \dots \mu_\ell}. \quad (117)$$

In the 14-moment approximation we start by substituting Eqs. (48) and (49) into Eq. (114) and obtain

$$C_{r-1} = C_{\Pi\chi} \Pi X_{r-3,1}, \quad (118)$$

$$C_{r-1}^{(\mu)} = \mathcal{B}_V V^\mu X_{r-2,3} + \mathcal{B}_W W^\mu X_{r-2,3}, \quad (119)$$

$$C_{r-1}^{(\mu\nu)} = \mathcal{A}_\pi \pi^{\mu\nu} X_{r-1,4}. \quad (120)$$

Here $X_{r,1}$, $X_{r,3}$ and $X_{r,4}$ are coefficients of the following rank-4 collision tensor,

$$X_r^{\mu\nu\alpha\beta} = \frac{1}{\nu} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \tilde{f}_{0\mathbf{p}} \tilde{f}_{0\mathbf{p}'} \times E_{\mathbf{k}}^r k^\mu k^\nu (p^\alpha p^\beta + p'^\alpha p'^\beta - k^\alpha k^\beta - k'^\alpha k'^\beta). \quad (121)$$

This collision tensor is symmetric upon the interchange of indices (μ, ν) and (α, β) , and also traceless for the latter indices,

$$X_r^{\mu\nu\alpha\beta} = X_r^{(\mu\nu)(\alpha\beta)}, \quad X_r^{\mu\nu\alpha\beta} g_{\alpha\beta} = 0. \quad (122)$$

These properties lead to a spatially isotropic tensor constructed using the four-velocity u^μ , the transverse projection $\Delta^{\mu\nu}$, and different scalar coefficients $X_{r,i}$ as

$$X_r^{\mu\nu\alpha\beta} = (X_{r,1} u^\mu u^\nu + X_{r,2} \Delta^{\mu\nu}) \left(u^\alpha u^\beta - \frac{1}{3} \Delta^{\alpha\beta} \right) + 4X_{r,3} u^{(\mu} \Delta^{\nu)(\alpha} u^{\beta)} + X_{r,4} \Delta^{\mu\nu\alpha\beta}, \quad (123)$$

where the coefficients of the above decomposition are obtained from the following contractions,

$$X_{r,1} = X_r^{\mu\nu\alpha\beta} u_\mu u_\nu u_\alpha u_\beta, \quad (124)$$

$$X_{r,2} = \frac{1}{3} X_r^{\mu\nu\alpha\beta} \Delta_{\mu\nu} u_\alpha u_\beta, \quad (125)$$

$$X_{r,3} = \frac{1}{3} X_r^{\mu\nu\alpha\beta} u_\mu \Delta_{\nu\alpha} u_\beta, \quad (126)$$

$$X_{r,4} = \frac{1}{5} X_r^{\mu\nu\alpha\beta} \Delta_{\mu\nu\alpha\beta}. \quad (127)$$

The evaluation of the coefficients $X_{r,i}$ requires the detailed knowledge of the differential cross section. As an example, these functions are evaluated for a classical gas of massless particles with constant cross section in Appendix A.

5.2 Equations of motion

In order to close the conservation laws Eqs. (1) and (2) we need additional equations for the dissipative currents which can be obtained from the exact equations of motion, Eqs. (67, 68, 69).

For any $r \geq 0$, Eq. (67) for the scalar moment leads to an equation of motion for the bulk viscous pressure. Replacing $\rho_r = \gamma_r^\Pi \Pi$ according to Eq. (101) in Eq. (67) and collecting all terms we obtain the so-called relaxation equation of the bulk viscous pressure,

$$\begin{aligned} \dot{\Pi} = & -\frac{\Pi}{\tau_\Pi^r} - \frac{\zeta^r}{\tau_\Pi^r} \theta + \tau_{\Pi W}^r W_\mu \dot{u}^\mu + \tau_{\Pi V}^r V_\mu \dot{u}^\mu \\ & - \ell_{\Pi W}^r \partial_\mu W^\mu - \ell_{\Pi V}^r \partial_\mu V^\mu - \delta_{\Pi \Pi}^r \Pi \theta \\ & + \lambda_{\Pi W}^r W^\mu \nabla_\mu \alpha_0 + \lambda_{\Pi V}^r V^\mu \nabla_\mu \alpha_0 + \lambda_{\Pi \pi}^r \pi^{\mu\nu} \sigma_{\mu\nu}. \end{aligned} \quad (128)$$

Here, we introduced the relaxation time of the bulk viscous pressure, τ_Π^r , and the bulk viscosity coefficient ζ^r as

$$\tau_\Pi^r = -\frac{\gamma_r^\Pi \mathcal{C}_\Pi^{-1}}{X_{r-3,1}}, \quad \zeta^r = -\tau_\Pi^r \frac{\alpha_r^{(0)}}{\gamma_r^\Pi}, \quad (129)$$

where \mathcal{C}_Π was defined in Eq. (110). Similarly, γ_r^Π was defined in Eq. (104), $X_{r,1}$ in Eq. (124), while $\alpha_r^{(0)}$ was given in Eq. (74). The other transport coefficients in Eq. (128) are defined as

$$\tau_{\Pi W}^r = \frac{1}{\gamma_r^\Pi} \left[(r-1) \gamma_{r-1}^W + \beta_0 \frac{\partial \gamma_{r-1}^W}{\partial \beta_0} + \frac{G_{2r}}{D_{20}} \right], \quad (130)$$

$$\tau_{\Pi V}^r = \frac{1}{\gamma_r^\Pi} \left[(r-1) \gamma_{r-1}^V + \beta_0 \frac{\partial \gamma_{r-1}^V}{\partial \beta_0} \right], \quad (131)$$

$$\lambda_{\Pi W}^r = -\frac{1}{\gamma_r^\Pi} \left(\frac{\partial \gamma_{r-1}^W}{\partial \alpha_0} + h_0^{-1} \frac{\partial \gamma_{r-1}^W}{\partial \beta_0} \right), \quad (132)$$

$$\lambda_{\Pi V}^r = -\frac{1}{\gamma_r^\Pi} \left(\frac{\partial \gamma_{r-1}^V}{\partial \alpha_0} + h_0^{-1} \frac{\partial \gamma_{r-1}^V}{\partial \beta_0} \right), \quad (133)$$

$$\ell_{\Pi W}^r = \frac{1}{\gamma_r^\Pi} \left(\gamma_{r-1}^W + \frac{G_{2r}}{D_{20}} \right), \quad (134)$$

$$\ell_{\Pi V}^r = \frac{1}{\gamma_r^\Pi} \left(\gamma_{r-1}^V - \frac{G_{3r}}{D_{20}} \right), \quad (135)$$

$$\lambda_{\Pi \pi}^r = \frac{1}{\gamma_r^\Pi} \left[(r-1) \gamma_{r-2}^\pi + \frac{G_{2r}}{D_{20}} \right], \quad (136)$$

$$\begin{aligned} \delta_{\Pi \Pi}^r = & \frac{n_0 D_{20}^{-1}}{\gamma_r^\Pi} \left[\left(J_{20} \frac{\partial \gamma_r^\Pi}{\partial \alpha_0} + J_{10} \frac{\partial \gamma_r^\Pi}{\partial \beta_0} \right) h_0 \right. \\ & - J_{30} \frac{\partial \gamma_r^\Pi}{\partial \alpha_0} - J_{20} \frac{\partial \gamma_r^\Pi}{\partial \beta_0} \left. \right] - \frac{1}{3\gamma_r^\Pi} \left[(r-1) m^2 \gamma_{r-2}^\Pi \right. \\ & \left. - (r+2) \gamma_r^\Pi - 3 \frac{G_{2r}}{D_{20}} \right]. \end{aligned} \quad (137)$$

Note that these coefficients are independent of the collision integral. We also point out that here we follow the notation of Refs. [25,26] for the coefficients. Furthermore the choice of the LRF eliminates terms involving either V^μ (for Eckart's choice) or W^μ (for Landau's choice).

In the very same manner we get a relaxation equation for both the particle diffusion current and the energy-momentum diffusion current. This equation follows from Eq. (68) using $\rho_r^\mu = \gamma_r^V V^\mu + \gamma_r^W W^\mu$, where rank-3 tensors $\rho_r^{\mu\nu\lambda}$ for any $r \geq 0$ were neglected. Thus, after some calculations we obtain

$$\begin{aligned} \dot{V}^\mu + \psi_r^W \dot{W}^\mu = & -\frac{V^\mu}{\tau_V^r} - \psi_r^W \frac{W^\mu}{\tau_W^r} + \frac{\kappa_q^r}{\tau_V^r \beta_0^2 h_0^2} \nabla^\mu \alpha_0 \\ & + \psi_r^W W_\nu \omega^{\mu\nu} + V_\nu \omega^{\mu\nu} \\ & - \psi_r^W \lambda_{WW}^r W_\nu \sigma^{\mu\nu} + \lambda_{VV}^r V_\nu \sigma^{\mu\nu} \\ & - \psi_r^W \delta_{WW}^r W^\mu \theta + \delta_{VV}^r V^\mu \theta \\ & - \tau_{q\Pi}^r \Pi \dot{u}^\mu - \tau_{q\pi}^r \pi^{\mu\nu} \dot{u}_\nu \\ & + \ell_{q\Pi}^r \nabla^\mu \Pi - \ell_{q\pi}^r \Delta_\alpha^\mu \partial_\nu \pi^{\alpha\nu} \\ & + \lambda_{q\Pi}^r \Pi \nabla^\mu \alpha_0 + \lambda_{q\pi}^r \pi^{\mu\nu} \nabla_\nu \alpha_0, \end{aligned} \quad (138)$$

where we defined the relaxation time of the particle diffusion current, τ_V^r , of the energy-momentum diffusion current, τ_W^r , and the heat conductivity coefficient κ_q^r ,

$$\tau_V^r = -\frac{\gamma_r^V}{\mathcal{B}_V X_{r-2,3}}, \quad (139)$$

$$\tau_W^r = -\frac{\gamma_r^W + \alpha_r^h}{\mathcal{B}_W X_{r-2,3}}, \quad (140)$$

$$\kappa_q^r = \tau_V^r \frac{\alpha_r^{(1)}}{\gamma_r^V} h_0^2 \beta_0^2. \quad (141)$$

Furthermore,

$$\psi_r^W = \frac{\gamma_r^W + \alpha_r^h}{\gamma_r^V}, \quad (142)$$

and \mathcal{B}_V , \mathcal{B}_W were defined in Eqs. (112), such that $\mathcal{B}_V = -h_0 \mathcal{B}_W$, while γ_r^V , γ_r^W were defined in Eqs. (105,106). In addition, $X_{r,3}$ was defined in Eq. (126), while $\alpha_r^{(1)}$ and α_r^h were given in Eqs. (75) and (77). The two relaxation times are not independent but are related to each other as $\tau_W^r = -h_0 \psi_r^W \tau_V^r$. The relaxation equation is written such that it is straightforward to rewrite it in either the Eckart or the Landau frame. Moreover, using the definition (21) of the heat flow it is clear that the above equation is a relaxation equation for this quantity.

The coefficients in Eq. (138) which only exist in either the Eckart or Landau frame are

$$\begin{aligned} \lambda_{WW}^r = & -\frac{1}{5\gamma_r^V \psi_r^W} \left[m^2 (2r-2) \gamma_{r-2}^W \right. \\ & \left. - (2r+3) \gamma_r^W - 5\alpha_r^h \right], \end{aligned} \quad (143)$$

$$\begin{aligned} \delta_{WW}^r = & \frac{n_0 D_{20}^{-1}}{\gamma_r^V \psi_r^W} \left[\left(J_{20} \frac{\partial \gamma_r^W}{\partial \alpha_0} + J_{10} \frac{\partial \gamma_r^W}{\partial \beta_0} \right) h_0 \right. \\ & - J_{30} \frac{\partial \gamma_r^W}{\partial \alpha_0} - J_{20} \frac{\partial \gamma_r^W}{\partial \beta_0} \left. \right] - \frac{1}{3\gamma_r^V \psi_r^W} \left[m^2 (r-1) \gamma_{r-2}^W \right. \\ & \left. - (r+3) \gamma_r^W - 4\alpha_r^h \right], \end{aligned} \quad (144)$$

and

$$\lambda_{VV}^r = \frac{1}{5\gamma_r^V} [m^2 (2r-2) \gamma_{r-2}^V - (2r+3) \gamma_r^V], \quad (145)$$

$$\begin{aligned} \delta_{VV}^r = & -\frac{n_0 D_{20}^{-1}}{\gamma_r^V} \left[\left(J_{20} \frac{\partial \gamma_r^V}{\partial \alpha_0} + J_{10} \frac{\partial \gamma_r^V}{\partial \beta_0} \right) h_0 \right. \\ & \left. - J_{30} \frac{\partial \gamma_r^V}{\partial \alpha_0} - J_{20} \frac{\partial \gamma_r^V}{\partial \beta_0} \right] \\ & + \frac{1}{3\gamma_r^V} [m^2 (r-1) \gamma_{r-2}^V V^\mu - (r+3) \gamma_r^V], \quad (146) \end{aligned}$$

while terms and coefficients which are not affected by either choice of the LRF are

$$\tau_{q\pi}^r = -\frac{1}{\gamma_r^V} \left[(r-1) \gamma_{r-1}^\pi + \beta_0 \frac{\partial \gamma_{r-1}^\pi}{\partial \beta_0} \right], \quad (147)$$

$$\begin{aligned} \tau_{q\Pi}^r = & -\frac{1}{3\gamma_r^V} [rm^2 \gamma_{r-1}^\Pi - (r+3) \gamma_{r+1}^\Pi \\ & - 3\alpha_r^h + m^2 \beta_0 \frac{\partial \gamma_{r-1}^\Pi}{\partial \beta_0} - \beta_0 \frac{\partial \gamma_{r+1}^\Pi}{\partial \beta_0}], \quad (148) \end{aligned}$$

$$\ell_{q\pi}^r = \frac{1}{\gamma_r^V} (\alpha_r^h + \gamma_{r-1}^\pi), \quad (149)$$

$$\ell_{q\Pi}^r = \frac{1}{\gamma_r^V} \left[\alpha_r^h - \frac{1}{3} (m^2 \gamma_{r-1}^\Pi - \gamma_{r+1}^\Pi) \right], \quad (150)$$

$$\lambda_{q\pi}^r = -\frac{1}{\gamma_r^V} \left(\frac{\partial \gamma_{r-1}^\pi}{\partial \alpha_0} + h_0^{-1} \frac{\partial \gamma_{r-1}^\pi}{\partial \beta_0} \right), \quad (151)$$

$$\begin{aligned} \lambda_{q\Pi}^r = & -\frac{1}{3\gamma_r^V} \left[m^2 \left(\frac{\partial \gamma_{r-1}^\Pi}{\partial \alpha_0} + h_0^{-1} \frac{\partial \gamma_{r-1}^\Pi}{\partial \beta_0} \right) \right. \\ & \left. - \left(\frac{\partial \gamma_{r+1}^\Pi}{\partial \alpha_0} + h_0^{-1} \frac{\partial \gamma_{r+1}^\Pi}{\partial \beta_0} \right) \right]. \quad (152) \end{aligned}$$

The relaxation equation of the shear-stress tensor follows from Eq. (69) by replacing $\rho_r^{\langle\mu\nu\rangle} = \gamma_r^\pi \pi^{\mu\nu}$, and neglecting rank-3 and rank-4 tensors, $\rho^{\mu\nu\lambda} = 0$ and $\rho^{\mu\nu\lambda\kappa} = 0$,

$$\begin{aligned} \dot{\pi}^{\langle\mu\nu\rangle} = & -\frac{\pi^{\mu\nu}}{\tau_\pi^r} + \frac{2\eta^r}{\tau_\pi^r} \sigma^{\mu\nu} + 2\lambda_{\pi\Pi}^r \Pi \sigma^{\mu\nu} \\ & + 2\tau_{\pi W}^r W^{\langle\mu} \dot{u}^{\nu\rangle} + 2\tau_{\pi V}^r V^{\langle\mu} \dot{u}^{\nu\rangle} \\ & + 2\ell_{\pi W}^r \nabla^{\langle\mu} W^{\nu\rangle} + 2\ell_{\pi V}^r \nabla^{\langle\mu} V^{\nu\rangle} \\ & - 2\lambda_{\pi W}^r W^{\langle\mu} \nabla^{\nu\rangle} \alpha_0 - 2\lambda_{\pi V}^r V^{\langle\mu} \nabla^{\nu\rangle} \alpha_0 \\ & - 2\lambda_{\pi\pi}^r \pi_\alpha^{\langle\mu} \sigma^{\nu\rangle\alpha} + 2\pi_\alpha^{\langle\mu} \omega^{\nu\rangle\alpha} - 2\delta_{\pi\pi}^r \pi^{\mu\nu} \theta, \quad (153) \end{aligned}$$

where we defined the relaxation time τ_π^r for the shear-stress tensor and the shear viscosity coefficient η^r ,

$$\tau_\pi^r = -\frac{\gamma_r^\pi}{\mathcal{A}_\pi X_{r-1,4}}, \quad \eta^r = \tau_\pi^r \frac{\alpha_r^{(2)}}{\gamma_r^\pi}. \quad (154)$$

Here, \mathcal{A}_π was given in Eq. (113), γ_r^π in Eq. (107), $X_{r,4}$ in Eq. (127), while $\alpha_r^{(2)}$ was quoted in Eq. (76). The other

coefficients in Eq. (153) are

$$\begin{aligned} \lambda_{\pi\Pi}^r = & \frac{1}{15\gamma_r^\pi} [(r-1) m^4 \gamma_{r-2}^\Pi - (2r+3) m^2 \gamma_r^\Pi \\ & + (r+4) \gamma_{r+2}^\Pi], \quad (155) \end{aligned}$$

$$\begin{aligned} \tau_{\pi W}^r = & \frac{1}{5\gamma_r^\pi} \left[rm^2 \gamma_{r-1}^W + m^2 \beta_0 \frac{\partial \gamma_{r-1}^W}{\partial \beta_0} \right. \\ & \left. - (r+5) \gamma_{r+1}^W - \beta_0 \frac{\partial \gamma_{r+1}^W}{\partial \beta_0} \right], \quad (156) \end{aligned}$$

$$\begin{aligned} \tau_{\pi V}^r = & \frac{1}{5\gamma_r^\pi} \left[rm^2 \gamma_{r-1}^V + m^2 \beta_0 \frac{\partial \gamma_{r-1}^V}{\partial \beta_0} \right. \\ & \left. - (r+5) \gamma_{r+1}^V - \beta_0 \frac{\partial \gamma_{r+1}^V}{\partial \beta_0} \right], \quad (157) \end{aligned}$$

$$\ell_{\pi W}^r = -\frac{1}{5\gamma_r^\pi} (m^2 \gamma_{r-1}^W - \gamma_{r+1}^W), \quad (158)$$

$$\ell_{\pi V}^r = -\frac{1}{5\gamma_r^\pi} (m^2 \gamma_{r-1}^V - \gamma_{r+1}^V), \quad (159)$$

$$\begin{aligned} \lambda_{\pi W}^r = & \frac{1}{5\gamma_r^\pi} \left[m^2 \left(\frac{\partial \gamma_{r-1}^W}{\partial \alpha_0} + h_0^{-1} \frac{\partial \gamma_{r-1}^W}{\partial \beta_0} \right) \right. \\ & \left. - \left(\frac{\partial \gamma_{r+1}^W}{\partial \alpha_0} + h_0^{-1} \frac{\partial \gamma_{r+1}^W}{\partial \beta_0} \right) \right], \quad (160) \end{aligned}$$

$$\begin{aligned} \lambda_{\pi V}^r = & \frac{1}{5\gamma_r^\pi} \left[m^2 \left(\frac{\partial \gamma_{r-1}^V}{\partial \alpha_0} + h_0^{-1} \frac{\partial \gamma_{r-1}^V}{\partial \beta_0} \right) \right. \\ & \left. - \left(\frac{\partial \gamma_{r+1}^V}{\partial \alpha_0} + h_0^{-1} \frac{\partial \gamma_{r+1}^V}{\partial \beta_0} \right) \right], \quad (161) \end{aligned}$$

$$\lambda_{\pi\pi}^r = -\frac{1}{7\gamma_r^\pi} [(2r-2) m^2 \gamma_{r-2}^\pi - (2r+5) \gamma_r^\pi], \quad (162)$$

$$\begin{aligned} \delta_{\pi\pi}^r = & \frac{n_0 D_{20}^{-1}}{2\gamma_r^\pi} \left[\left(J_{20} \frac{\partial \gamma_r^\pi}{\partial \alpha_0} + J_{10} \frac{\partial \gamma_r^\pi}{\partial \beta_0} \right) h_0 \right. \\ & \left. - J_{30} \frac{\partial \gamma_r^\pi}{\partial \alpha_0} - J_{20} \frac{\partial \gamma_r^\pi}{\partial \beta_0} \right] \\ & - \frac{1}{6\gamma_r^\pi} [(r-1) m^2 \gamma_{r-2}^\pi - (r+4) \gamma_r^\pi]. \quad (163) \end{aligned}$$

In the above relaxation equations we have expressed the proper-time derivative and spatial derivative of the coefficients from Eqs. (101, 102, 103) using Eqs. (71, 72, 73). Therefore, for any coefficient γ_r^Π , γ_r^V , γ_r^W , or γ_r^π , collectively denoted by $\psi(\alpha_0, \beta_0)$, we used the following formula for the proper time derivative

$$\begin{aligned} \dot{\psi} = & \frac{n_0}{D_{20}} \left[\left(J_{20} \frac{\partial \psi}{\partial \alpha_0} + J_{10} \frac{\partial \psi}{\partial \beta_0} \right) h_0 \right. \\ & \left. - \left(J_{30} \frac{\partial \psi}{\partial \alpha_0} + J_{20} \frac{\partial \psi}{\partial \beta_0} \right) \right] \theta, \quad (164) \end{aligned}$$

while for the spatial gradient of ψ we used

$$\nabla^\mu \psi = \left(\frac{\partial \psi}{\partial \alpha_0} + h_0^{-1} \frac{\partial \psi}{\partial \beta_0} \right) \nabla^\mu \alpha_0 - \beta_0 \frac{\partial \psi}{\partial \beta_0} \dot{u}^\mu. \quad (165)$$

Note that these two equations follow from the equations of ideal fluid dynamics and we neglected terms from the gen-

eral conservation equations proportional to the dissipative fields or their derivatives.

6 Choice of moment and coefficients in the massless limit

As was shown in the previous section, *once* the 14-moment approximation is applied, *any* moment of the Boltzmann equation will lead to a closed set of equations which looks formally the same, see Eqs. (128), (138), and (153). This is immediately apparent in these equations by the explicit dependence of the transport coefficients on the index r . Barring any miraculous cancellation, already at this point it is obvious that their values will in general be different for different r . Thus, the 14-moment approximation leads to an ambiguity, since it gives rise to an infinite set of equations to describe a finite set of macroscopic variables.

In order to make this clear, we shall consider two different choices for the moments of the Boltzmann equation: the first one is the traditional choice of Israel and Stewart [15] and the other one was recently proposed by Denicol, Koide, and Rischke (DKR) [16], following Grad's original idea.

6.1 Equations of motion of Israel and Stewart

Israel and Stewart assumed that the equations of motion for the dissipative currents could be extracted from the second moment of the Boltzmann equation [8,9,15],

$$\partial_\mu \langle k^\mu k^\nu k^\lambda \rangle = \int dK k^\nu k^\lambda C[f], \quad (166)$$

with the equations for Π , V^μ or W^μ , and $\pi^{\mu\nu}$ obtained using the following projections of the above equation,

$$u_\nu u_\lambda \partial_\mu \langle k^\mu k^\nu k^\lambda \rangle = u_\nu u_\lambda \int dK k^\nu k^\lambda C[f], \quad (167)$$

$$\Delta_\lambda^\alpha u_\nu \partial_\mu \langle k^\mu k^\nu k^\lambda \rangle = \Delta_\lambda^\alpha u_\nu \int dK k^\nu k^\lambda C[f], \quad (168)$$

$$\Delta_{\nu\lambda}^{\alpha\beta} \partial_\mu \langle k^\mu k^\nu k^\lambda \rangle = \Delta_{\nu\lambda}^{\alpha\beta} \int dK k^\nu k^\lambda C[f], \quad (169)$$

respectively, *together* with the 14-moment approximation.

As a matter of fact, Eqs. (167), (168) and (169) can be identified as the equations for ρ_3 , ρ_2^μ , and $\rho_1^{\mu\nu}$. These equations have already been calculated with the 14-moment approximation, Eqs. (128), (138), and (153). Thus, using the indices $r = 3$ (for the scalar moment), $r = 2$ (for the vector), and $r = 1$ (for the irreducible second-rank tensor), we obtain the IS equations.

Even though this choice of moments was never clearly justified, this prescription is widely employed in relativistic kinetic theory. However, it was recently found that, at least for some cases, the IS equations are not in good agreement with the numerical solution of the Boltzmann equation [18,19,20,21,22,23]. Also, the transport coefficients obtained by Israel and Stewart do not coincide with quantum-field theoretical calculations [24].

6.2 Equations of motion directly from the dissipative currents

In the second choice, the equations of motion for the dissipative currents are obtained from the moments ρ_0 , ρ_0^μ , and $\rho_0^{\mu\nu}$ which exactly correspond to the dissipative currents, see Eqs. (58), (59), and (61). Here, we have already fixed the LRF in accordance with the Landau picture. Then, the equations of motion for the dissipative currents emerge directly from their definitions as

$$\dot{\Pi} = -\frac{1}{3}m^2 \int dK \delta \dot{f}_{\mathbf{k}}, \quad (170)$$

$$\dot{V}^{\langle\mu\rangle} = \int dK k^{\langle\mu\rangle} \delta \dot{f}_{\mathbf{k}}, \quad (171)$$

$$\dot{\pi}^{\langle\mu\nu\rangle} = \int dK k^{\langle\mu} k^{\nu\rangle} \delta \dot{f}_{\mathbf{k}}. \quad (172)$$

As already mentioned, these equations are related to the equations for $\rho_0 = -3\Pi/m^2$, $\rho_0^\mu = V^\mu$, and $\rho_0^{\mu\nu} = \pi^{\mu\nu}$, that is, Eqs. (67), (68), and (69), for $r = 0$. In this scenario, the fluid-dynamical equations (1,2) are closed by Eqs. (128), (138), and (153), with $r = 0$, which correspond to Eqs. (170), (171), and (172), once the 14-moment approximation is applied.

It was shown that this method can successfully reproduce the numerical solution of the Boltzmann equation for the simple one-dimensional scaling expansion [16]. It is also important to mention that the transport coefficients of this kinetic calculation are consistent with those calculated from quantum field theory with the method proposed in Ref. [24].

6.3 Comparison of choices

In order to understand the difference between the two approaches discussed above, we calculate the coefficients $\beta_\Pi^r = -\zeta^r/\tau_\Pi^r$, $\beta_\pi^r = \eta^r/\tau_\pi^r$, and $\beta_V^r = \kappa_q^r/(\tau_V^r \beta_0^2 h_0^2)$. These coefficients, normalized by the pressure or particle density, are shown in Figs. 1, 2, and 3, respectively. The calculations were done for a classical gas with fixed chemical potential, $\mu = 0$.

We see that both calculations converge at low temperatures but deviate considerably at high temperatures. This behavior should be qualitatively the same for any choice of moment because all irreducible moments of the same rank converge to the same values in the non-relativistic limit (multiplied by a different power of the mass). Thus, differences between the choice of moment will only appear in the relativistic limit.

The coefficients in the ultrarelativistic limit, $m/T = 0$, for a classical gas with constant cross section, can be calculated analytically. These are collected for the shear viscosity and particle diffusion in Tables 1, 2, and 3. Note that, in this limit, the bulk viscous pressure vanishes and was not considered. For the relaxation times, τ_V and τ_π , and transport coefficients, η and κ_V , the differences are of the order of 10 – 20 %, but for other coefficients the differences can be more significant.

	η^r	τ_π^r	$\lambda_{\pi\pi}^r$	$\lambda_{\pi V}^r$	$\lambda_{\pi W}^r$	$\delta_{\pi\pi}^r$	$\ell_{\pi V}^r$	$\ell_{\pi W}^r$	$\tau_{\pi V}^r$	$\tau_{\pi W}^r$
$r = 1$ (IS)	$6\sigma_T^{-1}/(5\beta_0)$	$9\sigma_T^{-1}/(5n_0)$	1	$-1/(3\beta_0)$	$1/12$	$2/3$	$-2/(3\beta_0)$	$1/3$	$8/(3\beta_0)$	$-5/3$
$r = 0$ (DKR)	$4\sigma_T^{-1}/(3\beta_0)$	$5\sigma_T^{-1}/(3n_0)$	$5/7$	0	0	$2/3$	0	$1/5$	0	-1

Table 1. The coefficients for the shear-stress tensor in the two approaches for the classical gas with constant cross section in the ultrarelativistic limit.

	κ_q^r	τ_V^r	ψ_W^r	τ_W^r	δ_{VV}^r	δ_{WW}^r	λ_{VV}^r	λ_{WW}^r
$r = 2$ (IS)	$2\sigma_T^{-1}$	$5\sigma_T^{-1}/(2n_0)$	$-\beta_0/4$	$5\sigma_T^{-1}/(2n_0)$	-1	$-4/3$	$-7/5$	$9/5$
$r = 0$ (DKR)	$3\sigma_T^{-1}$	$9\sigma_T^{-1}/(4n_0)$	$-\beta_0/4$	$9\sigma_T^{-1}/(4n_0)$	-1	$-4/3$	$-3/5$	1

Table 2. The coefficients for the particle and energy diffusion in the two approaches for a classical gas with constant cross section in the ultrarelativistic limit.

	$\lambda_{V\pi}^r$	$\lambda_{W\pi}^r$	$\ell_{V\pi}^r$	$\ell_{W\pi}^r$	$\tau_{V\pi}^r$	$\tau_{W\pi}^r$
$r = 2$ (IS)	$-3\beta_0/40$	$3/10$	$-\beta_0/20$	$1/5$	0	0
$r = 0$ (DKR)	$-\beta_0/20$	$1/5$	$-\beta_0/20$	$1/5$	0	0

Table 3. The coefficients which couple shear stress and particle or energy diffusion in the two approaches for a classical gas with constant cross section in the ultrarelativistic limit.

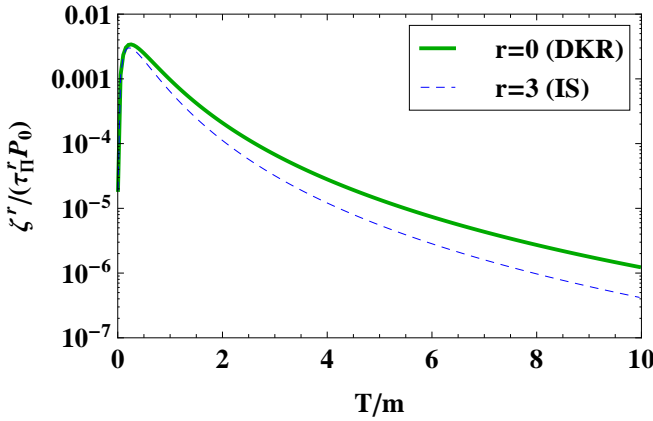


Fig. 1. The coefficient β_π^r normalized by the pressure P_0 . The cases $r = 3$ (dashed line) and $r = 0$ (solid) line correspond to the choices by Israel and Stewart and by Denicol, Koide and Rischke.

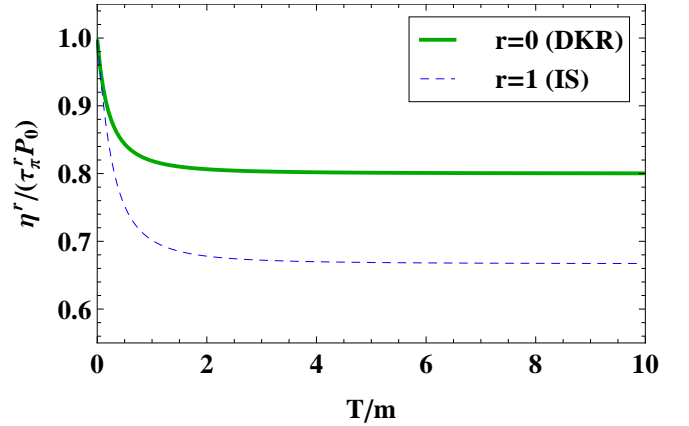


Fig. 2. The coefficient β_π^r normalized by the pressure P_0 . The cases $r = 1$ (dashed line) and $r = 0$ (solid) line correspond to the choices by Israel and Stewart and by Denicol, Koide and Rischke.

The coefficients $\lambda_{q\pi}^r$, $\ell_{q\pi}^r$, and $\tau_{q\pi}^r$ in Eq. (138) are calculated both in the Eckart and Landau frames, see Table 3. For example, in the Landau frame we only have an equation for V^μ and hence $\lambda_{V\pi}^r = \lambda_{q\pi}^r$ is given exactly by Eq. (151), while in the Eckart frame we only obtain an equation for W^μ so that, $\lambda_{\pi W}^r = \lambda_{\pi q}^r / \psi_W^r$. However, if we use the definition of the heat flow in either frame, i.e., $q^\mu = -h_0 V^\mu$ or $q^\mu = W^\mu$, then these coefficients lead to the same values for a classical gas where $\psi_r^W = -h_0^{-1}$.

7 Conclusions

In this work we have reviewed the 14-moment approximation proposed by Israel and Stewart and discussed the ambiguities of this approach. We started by introducing a

general expansion of the single-particle distribution function in terms of its moments. For this purpose, we constructed an orthonormal expansion basis which allowed us to establish exact relations between the expansion parameters and the moments of the distribution function. We then proceeded to derive the exact equations of motion for these moments.

Next, we showed how the 14-moment approximation can be obtained as a truncation of this general expansion of the distribution function. We proved that, once the 14-moment approximation has been applied, it is possible to derive an infinite number of fluid-dynamical equations, all having the same general structure but with different transport coefficients. This means that the 14-moment approximation is not able to provide a unique theory of fluid dynamics and, in this sense, is ambiguous. In Sec. 6 we

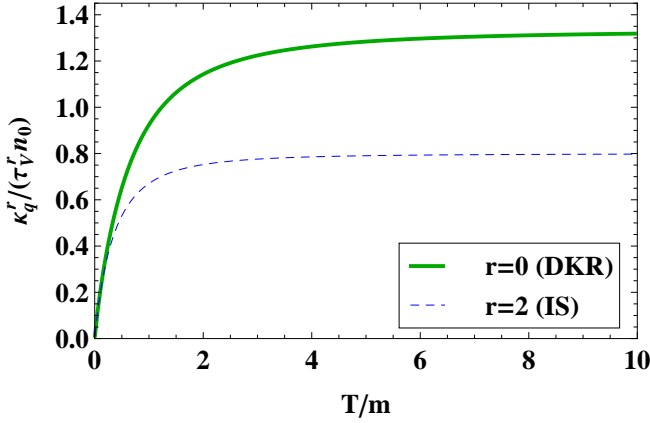


Fig. 3. The coefficient β_V^r normalized by the pressure P_0 . The cases $r = 2$ (dashed line) and $r = 0$ (solid line) correspond to the choices by Israel and Stewart and by Denicol, Koide and Rischke.

analysed two different choices for the moment equations: the one corresponding to Israel and Stewart [15], and the other one to that of Denicol, Koide, and Rischke [16]. It is also worth to mention that in this derivation we obtained terms that were neglected in the original work of IS [15], as was already presented in Ref. [25,26].

We also remark that the solutions of the IS equations were already compared to the numerical solutions of the Boltzmann equation for the so-called Bjorken-scaling problem in Refs. [18,19,20] and for the relativistic Riemann problem in Ref. [21,22,23]. It was demonstrated that IS theory is in relatively good agreement with the numerical solutions of the Boltzmann equation only if the Knudsen number is sufficiently small. Note that these comparisons did not include all non-linear terms and transport coefficients derived in this work. On the other hand, in Ref. [16] it was shown that, in contrast to IS theory, the direct method gives a much better agreement with the numerical solution of the Boltzmann equation up to very large Knudsen numbers.

Before closing we mention that recently the method presented in this work was extended to include $14 + 9 \times n$ moments. It was explicitly shown how to successively improve the expression for the transport coefficients by extending the number of moments from $n = 0$ to $n = 1, 2$, and 3 [17]. Furthermore, it was also shown that the equations of motion can be closed in terms of 14 dynamical variables without making use of the direct truncation of the moment expansion, the 14-moment approximation. This was obtained by a separation of the microscopic time scales and a power-counting scheme in Knudsen and inverse Reynolds number. The equations of motion can be closed in terms of only 14 dynamical variables, as long as we only keep terms of second order in Knudsen and/or inverse Reynolds number.

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A The collision integral in the massless limit

In this Appendix, we calculate the collision tensor defined in Eq. (121). For a classical gas with constant cross section,

$$X_r^{\mu\nu\alpha\beta} = \frac{1}{\nu} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \times E_{\mathbf{k}}^r k^\mu k^\nu (p^\alpha p^\beta + p'^\alpha p'^\beta - k^\alpha k^\beta - k'^\alpha k'^\beta). \quad (173)$$

First we define the total cross section as

$$\sigma_T(s) = \frac{1}{\nu} \int 2\pi \sin \Theta_s d\Theta_s \sigma(s, \Theta_s), \quad (174)$$

where $\sigma(s, \Theta_s)$ is the differential cross section, s is a collision invariant, i.e., a Mandelstam variable, and Θ_s is the scattering angle,

$$s \equiv (k^\mu + k'^\mu)^2 = (p^\mu + p'^\mu)^2, \quad (175)$$

$$\Theta_s = \arccos \left[\frac{(k^\mu - k'^\mu)(p_\mu - p'_\mu)}{(k^\mu - k'^\mu)^2} \right]. \quad (176)$$

The transition rate $W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'}$ is written in terms of the differential cross section as

$$W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'} = (2\pi)^6 s \sigma(s, \Theta_s) \delta^4(k^\mu + k'^\mu - p^\mu - p'^\mu). \quad (177)$$

In order to simplify the calculations we divide $X_r^{\mu\nu\alpha\beta}$ into gain and loss parts,

$$X_r^{\mu\nu\alpha\beta} = \mathcal{G}_r^{\mu\nu\alpha\beta} - \mathcal{L}_r^{\mu\nu\alpha\beta}, \quad (178)$$

where

$$\mathcal{G}_r^{\mu\nu\alpha\beta} = \frac{1}{\nu} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \times E_{\mathbf{k}}^r k^\mu k^\nu (p^\alpha p^\beta + p'^\alpha p'^\beta), \quad (179)$$

$$\mathcal{L}_r^{\mu\nu\alpha\beta} = \frac{1}{\nu} \int dK dK' dP dP' W_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'} f_{0\mathbf{k}} f_{0\mathbf{k}'} \times E_{\mathbf{k}}^r k^\mu k^\nu (k^\alpha k^\beta + k'^\alpha k'^\beta). \quad (180)$$

The tensor $\mathcal{L}_r^{\mu\nu\alpha\beta}$ can be directly integrated and written in terms of the total cross section,

$$\mathcal{L}_r^{\mu\nu\alpha\beta} = \frac{1}{2} \int dK dK' f_{0\mathbf{k}} f_{0\mathbf{k}'} \sigma_T(s) \sqrt{s(s - 4m^2)} \times E_{\mathbf{k}}^r k^\mu k^\nu (k^\alpha k^\beta + k'^\alpha k'^\beta), \quad (181)$$

where we used

$$\frac{1}{\nu} \int \frac{d^3\mathbf{p}}{p^0} \frac{d^3\mathbf{p}'}{p'^0} s \sigma(s, \Theta_s) \delta_{\mathbf{k}\mathbf{k}' \rightarrow \mathbf{p}\mathbf{p}'}^4 = \frac{1}{2} \sigma_T(s) \sqrt{s(s - 4m^2)}. \quad (182)$$

For the tensor $\mathcal{G}_r^{\mu\nu\alpha\beta}$ we first introduce the total momentum and corresponding projection orthogonal to it,

$$P_T^\mu = k^\mu + k'^\mu = p^\mu + p'^\mu, \quad (183)$$

$$\Delta_{P_T}^{\mu\nu} = g^{\mu\nu} - \frac{P_T^\mu P_T^\nu}{s}. \quad (184)$$

Now, the p -dependent part of the integral can be written as,

$$\begin{aligned} & \frac{1}{\nu} \int \frac{d^3\mathbf{p}}{p^0} \frac{d^3\mathbf{p}'}{p'^0} s \sigma(s, \Theta_s) p^\alpha p^\beta \delta_{kk' \rightarrow pp'}^4 \\ &= B_1 P_T^\alpha P_T^\beta + B_2 \Delta_{P_T}^{\alpha\beta}, \end{aligned} \quad (185)$$

where in the center-of-momentum frame in which $P_T^\mu = (\sqrt{s}, 0, 0, 0)$ and $\Delta_{P_T}^{\mu\nu} p_\mu p_\nu = -|\mathbf{p}|^2$, we obtain

$$B_1 = \frac{1}{4} \sigma_T(s) \sqrt{s(s-4m^2)}, \quad (186)$$

$$B_2 = -\frac{\sqrt{s}}{12} \sigma_T(s) \left(\sqrt{s-4m^2} \right)^3. \quad (187)$$

In the massless limit, the above results simplify considerably,

$$\begin{aligned} \mathcal{L}_r^{\mu\nu\alpha\beta} &= \frac{1}{2} \int dK dK' f_{0\mathbf{k}} f_{0\mathbf{k}'} s \sigma_T(s) \\ &\quad \times E_{\mathbf{k}}^r k^\mu k^\nu (k^\alpha k^\beta + k'^\alpha k'^\beta), \end{aligned} \quad (188)$$

$$\begin{aligned} \mathcal{G}_r^{\mu\nu\alpha\beta} &= \frac{1}{3} \int dK dK' f_{0\mathbf{k}} f_{0\mathbf{k}'} s \sigma_T(s) \\ &\quad E_{\mathbf{k}}^r k^\mu k^\nu \left(P_T^\alpha P_T^\beta - \frac{s}{4} g^{\alpha\beta} \right). \end{aligned} \quad (189)$$

From here on, we will consider only the case of constant cross section. Then, using $s \equiv 2(k^\mu k'_\mu) = 2(p^\mu p'_\mu)$, we directly obtain

$$\begin{aligned} \mathcal{L}_r^{\mu\nu\alpha\beta} &= \sigma_T \langle E_{\mathbf{k}}^r k^\mu k^\nu k^\alpha k^\beta k^\kappa \rangle_0 \langle k_\kappa \rangle_0 \\ &\quad + \sigma_T \langle E_{\mathbf{k}}^r k^\mu k^\nu k^\kappa \rangle_0 \langle k^\alpha k^\beta k_\kappa \rangle_0, \end{aligned} \quad (190)$$

and

$$\begin{aligned} \mathcal{G}_r^{\mu\nu\alpha\beta} &= \frac{2}{3} \sigma_T \langle E_{\mathbf{k}}^r k^\mu k^\nu k^\alpha k^\beta k^\kappa \rangle_0 \langle k_\kappa \rangle_0 \\ &\quad + \frac{4}{3} \sigma_T \langle E_{\mathbf{k}}^r k^\mu k^\nu k^\kappa k^{(\alpha} \rangle_0 \langle k^{\beta)} k_\kappa \rangle_0 \\ &\quad + \frac{2}{3} \sigma_T \langle E_{\mathbf{k}}^r k^\mu k^\nu k^\kappa \rangle_0 \langle k^\alpha k^\beta k_\kappa \rangle_0 \\ &\quad - \frac{1}{3} \sigma_T g^{\alpha\beta} \langle E_{\mathbf{k}}^r k^\mu k^\nu k^\kappa k^\lambda \rangle_0 \langle k_\kappa k_\lambda \rangle_0, \end{aligned} \quad (191)$$

Finally, using the definition of the thermodynamic integrals from Eq. (78) we obtain

$$\begin{aligned} X_r^{\mu\nu\alpha\beta} &= -\frac{\sigma_T}{3} I_{r+5}^{\mu\nu\alpha\beta\kappa} I_{1,\kappa} + \frac{4\sigma_T}{3} I_{r+4}^{\mu\nu\kappa(\alpha} I_{2,\kappa}^{\beta)} \\ &\quad - \frac{\sigma_T}{3} I_{r+3}^{\mu\nu\kappa} I_{3,\kappa}^{\alpha\beta} - \frac{\sigma_T}{3} g^{\alpha\beta} I_{r+4}^{\mu\nu\kappa\lambda} I_{2,\kappa\lambda}. \end{aligned} \quad (192)$$

Therefore, the different projections are given as

$$\begin{aligned} X_{r,1} &\equiv X_r^{\mu\nu\alpha\beta} u_\mu u_\nu u_\alpha u_\beta, \\ &= -\sigma_T \left[\frac{1}{3} (I_{r+5,0} I_{10} + I_{r+3,0} I_{30}) - 8 I_{r+4,1} I_{21} \right], \end{aligned} \quad (193)$$

$$\begin{aligned} X_{r,3} &\equiv \frac{1}{3} X_r^{\mu\nu\alpha\beta} u_\mu \Delta_{\nu\alpha} u_\beta \\ &= \frac{\sigma_T}{3} [I_{r+5,1} I_{10} - 4 I_{r+4,1} I_{21} - I_{r+3,1} I_{31}], \end{aligned} \quad (194)$$

and

$$\begin{aligned} X_{r,4} &\equiv X_r^{\mu\nu\alpha\beta} \Delta_{\mu\nu\alpha\beta} \\ &= -\frac{2\sigma_T}{3} [I_{r+5,2} I_{10} + 4 I_{r+4,2} I_{21}]. \end{aligned} \quad (195)$$

In order to calculate the coefficients in the massless Boltzmann limit, we use the following formula for the thermodynamic integrals

$$I_{n+r,q}(\alpha_0, \beta_0, m \rightarrow 0) = \frac{P_0 (r+n+1)!}{2\beta_0^{r+n-2} (2q+1)!!},$$

where $P_0 = g e^{\alpha_0} \beta_0^{-4} / \pi^2$, hence

$$X_{r,1} = -\frac{\sigma_T P_0^2 (r+4)!}{6\beta_0^{r+2}} (r^2 + 3r + 2), \quad (196)$$

$$X_{r,3} = \frac{\sigma_T P_0^2 (r+4)!}{18\beta_0^{r+2}} (r^2 + 7r + 6), \quad (197)$$

$$X_{r,4} = -\frac{\sigma_T P_0^2 (r+5)!}{45\beta_0^{r+2}} (r+10). \quad (198)$$

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